Math Problem of the Month - October 2018

We say that two integers are *relatively prime* if they have no common prime divisors. You may know that if two positive integers m and n are relatively prime, then 1 can be written as an integer linear combination of m and n. That is,

$$1 = ma + nb, \text{ for some choice of integers } a \text{ and } b.$$

For example, 4 and 13 are relatively prime, and $1 = 4(-3) + 13(1)$.

Notice that if $1 = 4a + 13b$ with a, b integers, necessarily one of a and b are negative. Surprisingly, however, when m and n are relatively prime *all but finitely many positive integers* can be written as *positive* integer linear combination.

Problem: First, show that there is a postive integer N such that for any integer k greater than or equal to N,

$$k = 4a + 13b, \text{ for some } \textit{nonnegative} \text{ integers } a \text{ and } b.$$

Now, **prove** the result hold for any two positive integers which are relatively prime.

Put your solution into the dropbox for the Math Problem of the Month!

Solution: We'll supply the proof for the general case.

Let m, n be relatively prime, and say $1 = am + bn$ for integers a, b with $a < 0$ and $b > 0$. Notice that for $1 \leq i \leq m-1$, $i = iam + ibn = -i|a|m + ibn$. Now, let $T > |a|(m-1)$. Then, for $1 \leq i \leq m-1$,

$$Tm + i = Tm + iam + ibn = (T - |a|i)m + ibn = (T + ia)m + ibn.$$

Since $T > -a(m-1) = |a|(m-1)$, **both** $T + ia$ and ib are positive. Also, clearly $Tm + m = (T+1)m$.

Thus, we set $N = Tm$. Then, for any natural number k

$$N + k = Tm + k = \begin{cases} (T+x)m, \text{ if } k = xm \\ (T+x+ia)m + ibn, \text{ if } k = xm + i, 1 \leq i \leq m-1. \end{cases}$$

Therefore, $N = Tm$ is a solution (though not minimal).