Math Problem of the Month - November 2018

You may remember from calculus that a convergent series may converge either absolutely or conditionally. This problem will illustrate the caution necessary when manipulating a series which converges conditionally.

Find a double-indexed sequence $(a_{n,m})_{n,m\in\mathbb{N}}$ that has the following four properties:

1. For all n, $R_n = \sum_{m=1}^{\infty} a_{n,m}$ exists.
2. For all m, $C_m = \sum_{n=1}^{\infty} a_{n,m}$ exists.
3. $\sum_{n=1}^{\infty} R_n = 0$.
4. $\sum_{m=1}^{\infty} C_n = \infty$.

Informally, you can view the sequence you found as entries in an $\mathbb{N} \times \mathbb{N}$ matrix. Properties 1 and 2 above say that the (infinite) sums across each row and column exist. Moreover, by Properties 3 and 4, your matrix has the curious property that if you sum first across rows, and then across columns you get 0, but if you sum first across columns, and then across rows you get ∞. This is noteworthy, since each term in both sums appears exactly once.

(Hint: It's probably easiest to think of the sequence as an infinite matrix. Since you need to get 0 one way and ∞ the other, you'll definitely need both positivie and negative entries.)

Put your solution into the dropbox for the Math Problem of the Month!

Solution: We'll make an infinite matrix where $R_n = 0$, $C_1 = 1$ and $C_m = \frac{1}{2}$ for all $m > 1$. Then,

$$\sum_{n=1}^{\infty} R_n = \sum_{n=1}^{\infty} 0 = 0, \text{ and } \sum_{m=1}^{\infty} C_m = 1 + \sum_{m=2}^{\infty} \frac{1}{2} = \infty.$$

Now we'll define the matrix.

$$\text{Let } (a_{n,m})_{n,m\in\mathbb{N}} = \begin{cases} 1, \text{ if } n = m \\ -\frac{1}{2}, \text{ if } m = 2n, \text{ or } m = 2n+1 \\ 0, \text{ otherwise.} \end{cases}$$

Here is the corresponding matrix.

$$\begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Notice that for each n, $R_n = \sum_{m=1}^{\infty} a_{n,m} = a_{n,n} + a_{n,2n} + a_{n,2n+1} = 1 - \frac{1}{2} - \frac{1}{2} = 0$, and $C_1 = \sum_{n=1}^{\infty} a_{n,1} = a_{1,1} = 1$. For m even, $m = 2k$, $C_m = \sum_{n=1}^{\infty} a_{n,m} = a_{k,m} + a_{m,m} = -\frac{1}{2} + 1 = \frac{1}{2}$. Similarly for $m = 2k+1$, $C_m = \sum_{n=1}^{\infty} a_{n,m} = a_{k,m} + a_{m,m} = -\frac{1}{2} + 1 = \frac{1}{2}$.