Math Problem of the Month - March 2019

In this month's problem, we'll explore one generalization of the infinite sum of a sequence. This will allow us to make sense of equations like

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + ... = \frac{1}{2}$$

Let's start. Let $(f_n)_{n=1}^{\infty}$ be a sequence and let $(S_n)_{n=1}^{\infty}$ be it's sequence of partial sums, so

$$S_1 = f_1, S_2 = f_1 + f_2, \text{ in general } S_n = \sum_{i=1}^{n} f_i.$$

Now, let $(\sigma_n)_{n=1}^{\infty}$ be defined by $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} S_i$.

We say $(f_n)_{n=1}^{\infty}$ Césaro sums to L if $\lim_{n\to\infty} \sigma_n = L$. One can show that if a series converges to L its sequence Césaro sums to L as well.

On the other hand, if

$(f_n)_{n=1}^{\infty} = 1, -1, 1, -1, 1, -1, 1, -1, ...,$ then $(S_n)_{n=1}^{\infty} = 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, ...$

and $(\sigma_n)_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{3}{6}, \frac{3}{7}, \frac{4}{8}...,$ so indeed $(f_n)_{n=1}^{\infty}$ Césaro sums to $\frac{1}{2}$.

Thus Césaro sums properly generalize standard infinite sums.

Now, for this month's problem find two sequences $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ such that:

i. $(a_n)_{n=1}^{\infty}$ Césaro sums to something, but its sequence of partial sums isn't bounded.

ii. $(b_n)_{n=1}^{\infty}$ doesn't Césaro sums to anything, but its sequence of partial sums is bounded.

Put your solution into the dropbox for the Math Problem of the Month!