Math Problem of the Month - February 2019

Sequences occur in all branches of Mathematics. You may know some famous recursive sequences like the Fibonacci numbers: $F_0 = 0, F_1 = 1$ and for $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$. This is an example of a **linear** recursive sequence. Less common examples are **nonlinear**.

We'll define a nonlinear recursive sequence.

First, let $W_0 = 0, W_1 = 1, W_2 = 1, W_3 = -3, W_4 = 1$.

Now, define the following recursion for all $m > n$,

$$W_{m+n}W_{m-n} = W_{m+1}W_{m-1}W_n^2 - W_{n+1}W_{n-1}W_m^2.$$

This turns out to produce a well-defined sequence (there's no need to prove this).

Now **prove** that for all $n \in \mathbb{N}$, $W_n \in \mathbb{Z}$. To start, you may wish to gain intuition by calculating W_5.

Put your solution into the dropbox for the Math Problem of the Month!

First, notice as a consequence of the recurrence, that since $W_1 = 1$, taking $m = n + 1$ gives: $W_{2n+1} = W_{2n+1}W_1 = W_{n+2}W_n^3 - W_{n+1}^3 W_{n-1}$, which proves inductively that the odd numbered elements are all integers.

Taking $m = n+2$, since $W_2 = 1$, we have $W_{2n+2} = W_{2n+2}W_2 = W_{n+3}W_{n+1}W_n^2 - W_{n+1}W_{n-1}W_{n+2}^2 = W_{n+1}(W_{n+3}W_n^2 - W_{n-1}W_{n+2}^2)$, and by induction, the right hand side is an integer.

This proves the assertion.