Convergence to Rhombic Tiling in Phyllotaxis: An Exploration in the Dynamical Systems of Plant Formation

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Abstract

Phyllotaxis is the study of pattern formation in plants. In nature, many plant species exhibit spiral patterns corresponding to Fibonacci numbers. This research looks at the formation of such patterns at the cellular level by simulating plant growth with the Mathematica program Sno and Draw developed by Chris Golé and Pau Atela. Plant patterns are formed by groups of cells, called primordia, at the tip of the plant. They are represented by disks, which by successive accretion form tilings on a cylindrical surface. The geometry of the tilings is determined by the angles of the vectors connecting the primordia centers of a front, and by perturbing these angles, rhombic, pentagonal, or triangular tilings are formed. Images of tilings generated by varying angle conditions were produced in Mathematica and an analysis of resulting tilings gave evidence of infinite and finite convergence to rhombic tilings. The tilings generated in Mathematica were then used in understanding Stéphane Douady's sketch of a proof (derived from work with Atela) that all tilings eventually converge to a rhombic pattern.

Background Information

The number of visible spirals (parastichies) in spiral arrangements is most often a Fibonacci number $(1, 1, 2, 3, 5, 8, 13, 21 \ldots)$ and the angle between successive leaves is close to the Golden Angle - about 137.5 degrees.

Figure 1: (From left to right) 8 parastiches, 13 parastiches, 21 parastiches, 34 parastiches

The visible spiral pattern in mature plants is a result of the formation of spiral lattices on cylindrical surfaces at the microscopic level.

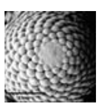

Figure 2: Norway Spruce exhibiting an $(8, 13)$ parastichy pattern

Witch's Hats and Bonnets

Definition: A witch's hat is a configuration in which exactly one up vector and one down vector are steeper than the other vectors connecting the primordia centers. A bonnet is a configuration in which exactly one up vector and one down vector are less steep than the other vectors.

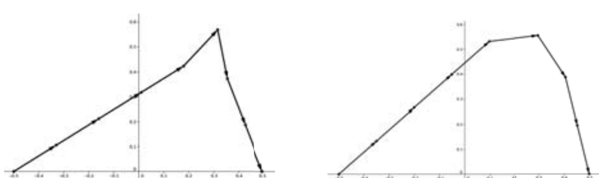

Figure 3: A $(5, 3)$ Witch's Hat and a $(5, 3)$ Bonnet.

Both the witch's hat and the bonnet generate pentagons and triangles aligned in the parastichies.

Example of Parastichies with Pentagons and Triangles

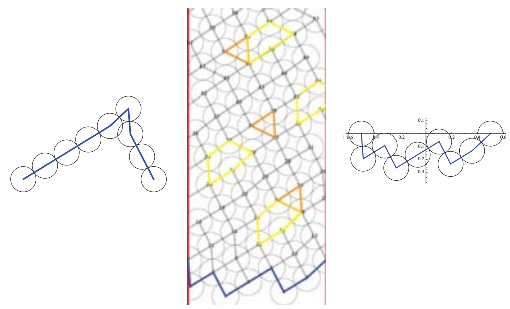

Figure 4: Tiling generated from a $(5, 3)$ witch's hat

Infinite Versus Finite Convergence Over Varying ϵ and δ

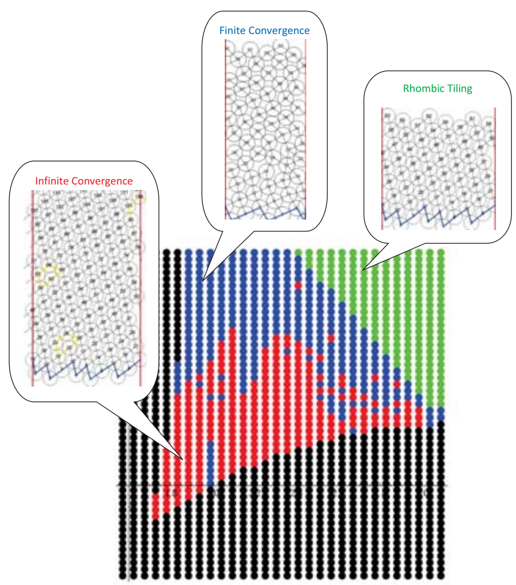

Figure 5: As ϵ and δ vary from 0 to π the pattern of convergence changes. The red dots converge to rhombic tiling in infinite time, the blue in finite, and the green start as rhombic tilings.

Examining Convergence in Bonnets

Let $\overrightarrow{OA}$ be a "normal" down vector and let $\overrightarrow{AB}$ be a perturbed down vector. The difference in the angles formed by these vectors will be denoted ϵ. Let $\overrightarrow{BE}$ be a perturbed up vector and ϕ' be the angle between this vector and the extension of $\overrightarrow{OA}$.

From this, a new primordium, C, will be formed tangentially to the parent primordia B and E. Let δ be the difference in the angles formed by the vector $\overrightarrow{CE}$ and the vector $\overrightarrow{AB}$.

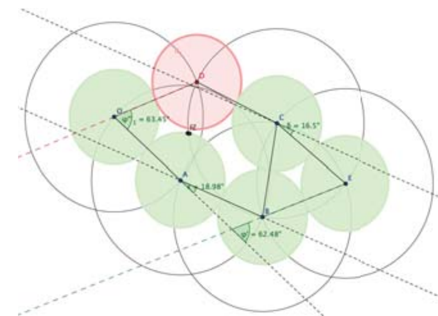

Figure 6: A pentagon and triangle pair from a bonnet configuration. The new primordium is red.

Conjecture: If $\epsilon \leq \delta$ then the configuration immediately stabilizes to rhombic tiling. Thus, we have convergence in finite time.

If instead $\epsilon > \delta$, then a pentagon will form in the parastichy. Let the angle that is formed between vector $\overrightarrow{OA}$ and the vector from the new primordia be denoted by ϕ'''.

Conjecture: $\phi''' \approx \phi'$, up to $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\delta^2)$.

From here there are two cases to consider:

- Case 1: $0 < 2\delta < \epsilon$,
- Case 2: $0 < 2\delta - \epsilon < \epsilon$.

Conjecture: If $0 < 2\delta < \epsilon$, then the configuration stabilizes to rhombic tiling. Thus, we have convergence in finite time.

Conjecture: If $0 < 2\delta - \epsilon < \epsilon$, then there will be new perturbation values $\epsilon' = \delta$ and $\delta' = 2\delta - \epsilon$. These new values will again fit in with Case 2. Thus, we have convergence in infinite time.

Future Work

In the future, we plan to:

- Prove that $\phi''' \approx \phi'$, up to $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\delta^2)$
- Examine Case 1 and Case 2 in more depth

Acknowledgements

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